IMAGES OF REAL REPRESENTATIONS OF $\mathrm{SL}_n(\mathbb{Z}_p)$

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ABSTRACT. In this paper, we investigate abstract homomorphism from the special linear group over complete discrete valuation rings with finite residue field, such as the ring of p-adic integers, into the general linear group over the reals. We find the minimal dimension in which such a representation has infinite image. For positive characteristic rings, this minimum is infinity.

1. INTRODUCTION

Borel and Tits showed in 1973 that in "most" cases, abstract homomorphisms between algebraic groups are in fact algebraic [BT73], i.e. *any* homomorphism $\varphi : G(k) \to G'(k')$ "almost" arises out of a field-morphism $k \to k'$.

In 1975 Margulis showed that higher rank lattices are superrigid. Employing the Borel-Harish Chandra theorems, this means that if $\mathbb{R}$ and k are a suitably chosen ring and field respectively then, any abstract homomorphism $G(\mathbb{R}) \to G'(k)$ again "almost" arises out of a ring-morphism $\mathbb{R} \to k$.

These results beg the following motivating question:

Question. *Let R and R' be rings and G and G' be group schemes so that $G(\mathbb{R})$ and $G'(\mathbb{R}')$ are well defined. When are the homomorphisms $G(\mathbb{R}) \to G'(\mathbb{R}')$ dictated by ring-morphisms $\mathbb{R} \to \mathbb{R}'$?*

This question has been addressed in several works of which we now give an overview.

Let $\mathrm{EL}_n(\mathbb{R})$ be the group generated by $n \times n$ elementary unipotent matrices with entries in the ring $\mathbb{R}$. As this is well defined for any ring, it provides us with an interesting class of examples to consider for the group G in our question. Answering questions along these lines, we have:

- [BT73] Let k be an infinite field, G and G' be absolutely almost simple algebraic groups with G simply connected or G' adjoint, and G generated by k-unipotents. Modulo the finite centers of G and G', any abstract

homomorphism $G(k) \to G'(k')$ with Zariski-dense image arises out of a field homomorphism $k \to k'$.

- [Mar91], [BHC61], [BHC62] Let $\mathcal{O}$ be the ring of integers of a number field k and G be higher rank and defined over k. Let $G'(\mathbb{C})$ be non compact. Then, any Zariski-dense homomorphism $G(\mathcal{O}) \to G'(\mathbb{C})$ arises from a ring-morphism $\mathcal{O} \to \mathbb{C}$.

- [Fer06] Let $n \geqslant 3$. Every homomorphism $\mathrm{SL}_n(\mathbb{Z}[x]) \to \mathrm{GL}_D \overline{\mathbb{Q}}$ is not injective. (Recall that $\mathrm{EL}_n(\mathbb{Z}[x]) = \mathrm{SL}_n(\mathbb{Z}[x])$.)

- [DG91] Let $n \geqslant 3$. Any semisimple representation $\mathrm{SL}_n(\mathbb{Z}[x_1, \ldots, x_m]) \to \mathrm{SL}_D \mathbb{C}$ is virtually the direct sum of tensor products of ring homomorphisms $\mathbb{Z}[x_1, \ldots, x_m] \to \mathbb{C}$.

- [KS09] Let $\mathbb{Z} \langle x, y \rangle$ be the free non-commutative ring on x and y. The group $\mathrm{EL}_3(\mathbb{Z} \langle x, y \rangle)$ does not have a faithful finite dimensional representation over any field.

- The most recent result is due to Igor Rapinchuk [Rap10]. It applies to the very general context of higher rank universal Chevalley-Demazure group schemes, describing their abstract representations into $\mathrm{GL}_D \mathbb{K}$, where $\mathbb{K}$ is an algebraically closed field. We state an example which we feel both captures the essence of the result and is relevant to our current work. Let $\mathfrak{O}$ be a local principal ideal ring and $n \geqslant 3$. Let $\varphi : \mathrm{SL}_n \mathfrak{O} \to \mathrm{GL}_D \mathbb{C}$ be an abstract homomorphism. If the image is not finite then there exists a commutative $\mathbb{C}$-algebra B, an embedding $\iota : \mathrm{SL}_n \mathfrak{O} \to \mathrm{SL}_n \mathbb{C}$ (induced from a ring embedding $\mathfrak{O} \to B$) so that, up to finite index, φ factors through ι composed with a $\mathbb{C}$-algebraic map $\mathrm{SL}_n \mathbb{C} \to \mathrm{GL}_D \mathbb{C}$.

  Remark: The general nature of Rapinchuk's result make us believe that, with some work, our result for $n \geqslant 3$ may be deduced from his. On the other hand, the elementary nature of our proof makes it of interest in its own right.

We remark that many (though not all) of the results above use the fact that the target group is with entries in an algebraically closed field.

Let $\mathfrak{O}$ be a complete discrete valuation ring with finite residue field, of cardinality p^β. It is well known that if $\mathfrak{O}$ is of characteristic 0, such as $\mathbb{Z}_p$, then $\mathfrak{O} \hookrightarrow \mathbb{C}$ which yields an abstract embedding $\mathrm{SL}_n \mathfrak{O} \hookrightarrow \mathrm{SL}_n \mathbb{C} \hookrightarrow \mathrm{SL}_{2n} \mathbb{R}$. In this paper, we investigate the representations $\mathrm{SL}_n \mathfrak{O} \to \mathrm{GL}_D \mathbb{R}$ where $D < 2n$. Let $\mathfrak{O}$ be a complete discrete valuation ring with finite residue field, of cardinality p^β. Our main result is the following:

Theorem 1.1. *For every n and any abstract homomorphism $\varphi : \mathrm{SL}_n\mathfrak{O} \to \mathrm{GL}_D\mathbb{R}$, where $D < 2n$ the image of φ is finite. If furthermore, $\mathfrak{O}$ has positive characteristic then the same is true for all D.*

As a direct consequence we obtain the following result about the special linear group $\mathrm{SL}_n(\mathbb{Z}_p)$, where $\mathbb{Z}_p$ is the ring of p-adic integers.

Corollary 1.2. *Let p be a rational prime. For every n and any homomorphism $\varphi : \mathrm{SL}_n\mathbb{Z}_p \to \mathrm{GL}_D\mathbb{R}$, where $D < 2n$ the image of φ is finite.*

Another consequence of our result is that if $D < 2n$ then the D-dimensional real representations of $SL_n(\mathfrak{O})$, as an abstract group, are continuous in the local-topology. It is known that the continuous complex representations of these groups always have finite image, and hence they factor through congruence quotients. However, little more is known. For some results on this see Nobs-Wolfart [NW74], Kutzko [Kut73], Avni-Klopsch-Onn-Voll [AKOV09] and [Sin].

In a different direction, there is an interesting conjecture which can be found in [Nik09] where it is attributed to Y. Barnea, E. Breullard, P.E. Caprace, T. Gelander and J. Wilson:

Conjecture (Blaubeuren). *A profinite group cannot have an infinite finitely generated image.*

Nikolov addresses this question for the class of "non-universal" groups, which contains the class of prosolvable groups. Nikolov outlines how a result of Segal [Seg00] gives that the conjecture is true for prosolvable groups and then goes on to prove the conjecture for non-universal groups [Nik09].

2. Outline of the Proof

The proof in non-zero characteristic is short and uninvolved. We outline here the inductive proof in zero characteristic.

Step 1: (Base case) We assume that the image of an abstract homomorphism $\mathrm{SL}_2\mathfrak{O} \to \mathrm{GL}_2\mathbb{R}$ is infinite and derive a contradiction. The assumption leads to

the existence of additive and multiplicative maps $(\mathfrak{O}, +) \to (\mathbb{R}, +)$ and $(\mathfrak{O}^*, \times) \to (\mathbb{R}^*, \times)$. Inspired by the lack of ring-morphisms $\mathfrak{O} \to \mathbb{R}$, we show that these maps are sufficiently compatible to get a contradiction. We then proceed to show that any representation $\mathrm{SL}_2\mathfrak{O} \to \mathrm{GL}_3\mathbb{R}$ is reducible.

Step 2: (Inductive step) We assume that the image of any homomorphism $\mathrm{SL}_{n-1}\mathfrak{O} \to \mathrm{GL}_{2n-3}\mathbb{R}$ is finite. We then embed $\mathrm{SL}_{n-1}\mathfrak{O}$ as a subgroup in $\mathrm{SL}_n\mathfrak{O}$ and by Corollary 4.7 we obtain that any homomorphism $\mathrm{SL}_n\mathfrak{O} \to \mathrm{GL}_D\mathbb{R}$ for $D \leqslant 2n - 3$ has finite image.

Step 3: (For $D = 2n - 2$ and $D = 2n - 1$) These cases will be obtained by generalizing the ideas given in the base case coupled with the inductive step.

3. Proof of the main theorem 1.2

Before we begin the proof, we discuss the main tools in proving the main theorem. The proofs of these and other algebraic facts are included in Section 4 for the convenience of the reader.

The guiding concept behind our theorem is of course the absense of unital ring-morphisms from $\mathfrak{O} \to \mathbb{R}$. Hence our objective is to show that a sufficient amount of the ring structure can be expressed in terms of the group structure of SL_n. Consider the following two equations:

$$[E_{1,2}(x), E_{2,3}(y)] = E_{1,3}(xy)$$

$$E_{1,3}(x) \cdot E_{1,3}(y) = E_{1,3}(x + y)$$

This shows that both the additive and multiplicative structures of a ring are embedded in the group structure of SL_n, at least if $n \geqslant 3$. This is not possible for $n = 2$ but there is still a sufficient amount of information that is held about the ring inside the group structure of SL_2, provided the ring has many units. The task is then to pass this information, via the homomorphism from the source to the target, which is the essence of the proof.

In both of these examples ($n \geqslant 3$ and $n = 2$) the information about the ring, is in fact carried by the minimal parabolic subgroup. To this end, we look toward understanding the images of N, the upper triangular subgroup of $\mathrm{SL}_n\mathfrak{O}$ and U the unipotent subgroup of N. The following is of great use:

Lemma 3.1. *Let $\varphi : N_n(\mathfrak{O}) \to \mathrm{GL}_D\mathbb{R}$ be a homomorphism. Then, there exists a normal finite index subgroup N_0 of N such that $U_0 = [N_0, N_0] \cap U$ is of finite index in U and so that the image $\varphi(U_0)$ is unipotent.*

This lemma is really a combination of Lemmas 4.9 and 4.10 in Section 4. Observe that $U_0 \lhd N$ and therefore, the maximal 1-eigen space of $\varphi(U_0)$ is $\varphi(N)$-invariant.

There are two more key ingredients. The first ingredient is that the image of $\mathrm{SL}_n\mathfrak{O}$ will be finite if the image of some elementary unipotent subgroup $E_{i,j}(\mathfrak{O})$ is finite (Corollary 4.7).

The second tool we use is the fact that the group $\mathrm{SL}_n\mathfrak{O}$ is strongly almost perfect, that is to say, every finite index subgroup has finite abelianization. In particular, if either $n \neq 2$ or $\mathfrak{O} \neq \mathbb{Z}_2, \mathbb{Z}_3$, we have that $\mathrm{SL}_n\mathfrak{O}$ is perfect, i.e. it has trivial abelianization (Proposition 4.8). So if the image of a finite index subgroup of $\mathrm{SL}_n(\mathfrak{O})$ is abelian then the image of $\mathrm{SL}_n(\mathfrak{O})$ is finite.

3.1. **Proof in positive characteristic.** Let $\mathfrak{O}$ be a ring of positive characteristic such that $\mathrm{SL}_n(\mathfrak{O})$ is generated by elementary unipotents, for example if $\mathfrak{O}$ is a local principal ideal ring (Lemma 4.4). Also let $\varphi : \mathrm{SL}_n\mathfrak{O} \to \mathrm{SL}_D\mathbb{R}$ be a homomorphism. We shall show that the image of φ is finite.

With Lemma 3.1 we find a finite index subgroup $U_0 \leqslant U$ so that $\varphi(U_0)$ is unipotent. But of course, as $\mathfrak{O}$ has positive characteristic, all the elements in U_0 have finite order which means that $\varphi(U_0)$ is finite and therefore the image $\varphi(\mathrm{SL}_n\mathfrak{O})$ is finite by Corollary 4.7.

3.2. **Proof in Characteristic Zero.** Assume that $\mathfrak{O}$ is a local principal ideal ring of characteristic zero.

Our proof relies on an understanding of the 1-eigenspaces in the image of certain subgroups. We will need the following:

Definition 3.2. *Let $S \leqslant \mathrm{GL}_n K$ be a solvable group where K is of characteristic 0 and S_0 be the maximal normal finite index subgroup so that $S_U = [S_0, S_0]$ is unipotent. The JH series (for Jordan-Holder) for S is defined as follows: $V_0 \subset V_1 \subset \cdots \subset V_e$ where V_j/V_{j-1} is the maximal 1-eigen space of S_U acting on K^n/V_{j-1}.*

We make a few remarks about the above definition. Observe that by the Lie-Kolchin theorem the decomposition is well defined. And as $S_U \lhd S$ we have that S, or more generally, the normalizer of S_U, preserves the series and if we restrict the action to $\bigoplus_{j=0}^{e} V_j/V_{j-1}$ then S_0 is abelian and semi-simple.

Also by extending a choice of basis for V_{j-1} to a basis for V_j we get what we will refer to as a JH-decomposition for our group S. We note that since the spaces are 1-eigen spaces for S_U, there is a matrix with coefficients in $\mathbb{R}$ which conjugates S into upper-block-triangular form.

3.2.1. *Step 1.1:* $\mathrm{SL}_2\mathfrak{O} \to \mathrm{GL}_2\mathbb{R}$. The content of Corollary 4.7 is that to show the image of such a representation is finite, it is sufficient to know that the image of $E_{i,j}(\mathbb{Z}_p)$ is finite, for some i, j. Hence, the proof in this base case follows from the following statement. Let N be the upper triangular group in $\mathrm{SL}_2\mathfrak{O}$ and U its maximal unipotent subgroup.

Proposition 3.3. *For any representation* $\varphi : N \to \mathrm{GL}_2\mathbb{R}$ *the image* $\varphi(U)$ *is finite.*

Proof. With the representation fixed, let $U_0 \leqslant U$ be the finite index subgroup guaranteed by Lemma 3.1 so that $\varphi(U_0)$ is unipotent.

If the 1-eigen space of $\varphi(U_0)$ is 2-dimensional then the result follows. Therefore, assume by contradiction that it is one dimensional. Since the image $\varphi(U_0)$ has $\mathbb{R}$-entries, the 1-eigen space is defined over $\mathbb{R}$ and so, up to post composing with an inner automorphism of $\mathrm{GL}_2\mathbb{R}$ we may assume that the image $\varphi(U_0)$ is upper triangular unipotent.

Since the image of the centralizer (respectively normalizer) of U_0 must centralize (respectively normalize) the image of U_0 we see that the image $\varphi(U)$ is upper triangular with ± 1 on the diagonal (and respectively the image $\varphi(N)$ is upper triangular).

This gives rise to an additive map $\psi_A : \mathfrak{O} \to \mathbb{R}$ and multiplicative maps $\psi_i : \mathfrak{O}^* \to \mathbb{R}^*$ as follows:

$$\varphi \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} \pm 1 & \psi_A(x) \\ 0 & \pm 1 \end{pmatrix}.$$

and

$$\varphi \begin{pmatrix} q & 0 \\ 0 & q^{-1} \end{pmatrix} = \begin{pmatrix} \psi_1(q) & * \\ 0 & \psi_2(q) \end{pmatrix} = \begin{pmatrix} \psi_1(q) & 0 \\ 0 & \psi_2(q) \end{pmatrix} \begin{pmatrix} 1 & * \\ 0 & 1 \end{pmatrix}$$

Consider the following relation for $q^2 \in \mathfrak{O}^*$, $x \in \mathfrak{O}$, $r \in \mathbb{Z}$:

$$\begin{pmatrix} q^2 & 0 \\ 0 & q^{-2} \end{pmatrix} \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} \begin{pmatrix} q^{-2} & 0 \\ 0 & q^2 \end{pmatrix} = \begin{pmatrix} 1 & q^4 x \\ 0 & 1 \end{pmatrix}$$

Using our definitions of ψ_i and ψ_A, after applying φ to both sides of the equation above and observing that $\begin{pmatrix} 1 & * \\ 0 & 1 \end{pmatrix}$ centralizes the image of U_0 we get the following:

$$\begin{pmatrix} \psi_1(q)^2 & * \\ 0 & \psi_2(q)^{-2} \end{pmatrix} \begin{pmatrix} \pm 1 & \psi_A(x) \\ 0 & \pm 1 \end{pmatrix} \begin{pmatrix} \psi_1(q)^2 & * \\ 0 & \psi_2(q)^{-2} \end{pmatrix} = \begin{pmatrix} \pm 1 & \psi_A(q^4 x) \\ 0 & \pm 1 \end{pmatrix}.$$

Performing the matrix multiplication, we obtain the following equation, which holds for every $x \in \mathbb{Z}_p$ and $q \in \mathbb{Z}_p^*$:

(1) $$\psi_1(q)^2 \psi_2(q)^2 \psi_A(x) = \psi_A(q^4 x).$$

By Lemma 4.3, we can find $q \in \mathfrak{O}^*$ so that q^4 is a negative integer, say $-r$. Using the fact that additive maps between abelian groups are $\mathbb{Z}$-equivariant, equation (1) becomes

$$(\psi_1(q)^2 \psi_2(q)^2 + r)\psi_A(x) = 0.$$

But the above expression is in $\mathbb{R}$ so that $\psi_1(q)^2 \psi_2(q)^2 + r$ must be positive. Therefore we must have $\psi_A(x) = 0$ for all $x \in \mathfrak{O}$. This contradicts our assumption that the 1-eigen space of $\varphi(U_0)$ is 1 dimensional. ✻

3.2.2. *Step 1.2 $\varphi : \mathrm{SL}_2\mathfrak{O} \to \mathrm{GL}_3\mathbb{R}$.*

Proof. We begin by giving the proof in case φ is reducible. We then show that any representation into $\mathrm{GL}_3\mathbb{R}$ must be reducible.

If φ is reducible, then there is an invariant subspace V of dimension 1 or 2. By looking at the restriction representation to $\mathrm{GL}(V)$ and the quotient representation to $\mathrm{GL}(\mathbb{R}^3/V)$ we get a map from the image $\varphi(\mathrm{SL}_2\mathfrak{O}) \to \mathrm{GL}_1\mathbb{R} \times \mathrm{GL}_2\mathbb{R}$ with abelian kernel. Applying the previously established fact that any representation $\mathrm{SL}_2\mathfrak{O} \to \mathrm{GL}_2\mathbb{R}$ has finite image, we see that $\varphi(\mathrm{SL}_2\mathfrak{O})$ has to be virtually abelian. But, as $\mathrm{SL}_2\mathfrak{O}$ is strongly almost perfect, we deduce that $\varphi(\mathrm{SL}_2\mathfrak{O})$ is finite.

We now show that either φ is reducible or has finite image. As before, we apply Lemma 3.1 to find U_0 of finite index in U so that $\varphi(U_0)$ is unipotent.

Let $V_1 \subset \mathbb{R}^3$ be the 1-eigen space of $\varphi(U_0)$. Recall that it is N invariant since $U_0 \trianglelefteq N$. If V_1 is a 3-dimensional space then image of U_0 is trivial and hence by Corollary 4.7, we get that the image of $SL_2\mathfrak{O}$ is finite. If V_1 is not 3-dimensional, then either V_1 or $\mathbb{R}^3/V_1$ is two dimensional.

Again, since V_1 is N-invariant, we get two homomorphisms $N \to \mathrm{GL}(V_1)$ and $N \to \mathrm{GL}(\mathbb{R}^3/V_1)$. By Proposition 3.3, we must have that the image of U in each is finite. In particular, by choice of V_1 the image of U_0 in both $\mathrm{GL}(V_1)$ and $\mathrm{GL}(\mathbb{R}^3/V_1)$ is trivial.

Therefore, up to post-composing φ with the transpose inverse automorphism of $\mathrm{GL}_3\mathbb{R}$ if necessary, we may assume that the 1-eigen space of $\varphi(U_0)$ has dimension two.

Now, since U_0 and U_0^t are conjugate inside $\mathrm{SL}_2\mathfrak{O}$, the 1-eigen space of the image $\varphi(U_0^t)$ has dimension two as well. Therefore the intersection of these two 2-dimensional spaces must be non-trivial in $\mathbb{R}^3$ which means that the image of the group $\langle U_0, U_0^t \rangle$ has a non-trivial 1-eigen space.

The group $\langle U_0, U_0^t \rangle$ is of finite index in $\mathrm{SL}_2\mathfrak{O}$. Up to passing to a further finite index subgroup if necessary, we may assume that it is normal in $\mathrm{SL}_2\mathfrak{O}$ and hence the non-trivial 1-eigen space of this finite index normal subgroup is invariant under $\mathrm{SL}_2\mathfrak{O}$. This means that φ is reducible.

❀

3.2.3. *Step 2:* In this section, whenever we speak of $\mathrm{SL}_{n-1}(\mathfrak{O}) \leqslant \mathrm{SL}_n(\mathfrak{O})$ we shall mean that it is the embedding into the upper left-hand corner of $\mathrm{SL}_n(\mathfrak{O})$.

To proceed with induction, we assume that the image of any homomorphism $\mathrm{SL}_{n-1}(\mathfrak{O}) \to \mathrm{GL}_{2n-3}(\mathbb{R})$ is finite. By considering $\mathrm{SL}_{n-1}(\mathfrak{O}) \leqslant \mathrm{SL}_n(\mathfrak{O})$ and using Corollary 4.7 we get that $\mathrm{SL}_n\mathfrak{O} \to \mathrm{GL}_D\mathbb{R}$ has finite image for all $D < 2n - 3$.

We are left to prove that the image of $\varphi : \mathrm{SL}_n(\mathfrak{O}) \to \mathrm{GL}_D(\mathbb{R})$ is finite if $2n - 3 < D < 2n$. As before, take U_0 so that its image is unipotent. Let L be the abelian subgroup of $\mathrm{SL}_n(\mathfrak{O})$ which has non-trivial entries in the last column. It is easily verified that L is normalized by $\mathrm{SL}_{n-1}(\mathfrak{O}) \leqslant \mathrm{SL}_n(\mathfrak{O})$. By intersecting L with U_0, we obtain a finite index subgroup L_0 of L whose image is unipotent. By Lemma 4.1, we can pass to a further finite index subgroup and assume that $L_0 = \pi^m L$ and is hence also normalized by $\mathrm{SL}_{n-1}(\mathfrak{O})$.

Now, since the image of L_0 is unipotent, we may find what we call a JH-series for $\varphi(L_0)$ (see Definition 3.2 and the remarks that follow). This is a flag $\{0\} = V_0 \subset V_1 \subset V_2 \subset \cdots \subset V_k = \mathbb{R}^D$ with the property that V_1 is the maximal 1-eigen space for $\varphi(L_0)$ and V_j is the maximal 1-eigen space for the quotient action on $\mathbb{R}^D/V_{j-1}$. Since $\varphi(L_0)$ is normalized by $\varphi(\mathrm{SL}_{n-1}(\mathfrak{O}))$, the JH-series of $\varphi(L_0)$ is preserved by $\varphi(\mathrm{SL}_{n-1}(\mathfrak{O}))$.

If $k = 1$, then $\mathbb{R}^D$ is the 1-eigenspace of $\varphi(L_0)$, that is to say the image of L_0 is trivial. Therefore the image of $E_{1,n}(\mathfrak{O}) \leqslant L$ is finite and again, Corollary 4.7 shows that the full image of $\mathrm{SL}_n\mathfrak{O}$ is finite.

We now assume that $k > 1$. The argument proceeds in two cases according to whether $2 \leqslant \dim(V_j) \leqslant D - 2$ for some j or not.

Assume that $2 \leqslant \dim(V_j) \leqslant D - 2$ for some j. By considering the action of $\varphi(\mathrm{SL}_{n-1}(\mathfrak{O}))$ on both V_j and $\mathbb{R}^D/V_j$ and applying the induction hypothesis, we get that the image of the map from $\varphi(\mathrm{SL}_{n-1}(\mathfrak{O}))$ to $\mathrm{GL}(V_j) \times \mathrm{GL}(\mathbb{R}^D/V_j)$ is finite. Let $\Gamma \leqslant \mathrm{SL}_{n-1}(\mathfrak{O})$ be the finite index subgroup so that its image

in $\mathrm{GL}(V_j) \times \mathrm{GL}(\mathbb{R}^D/V_j)$ is trivial. Since the kernel of the map $\mathrm{stab}(V_j) \to \mathrm{GL}(V_j) \times \mathrm{GL}(\mathbb{R}^D/V_j)$ is abelian, we see that $\varphi(\Gamma)$ is abelian, and hence finite since $\mathrm{SL}_{n-1}(\mathfrak{O})$ is strongly almost perfect. In particular, this implies that the image of $E_{12}(\mathfrak{O})$ is finite, which concludes the proof in this case.

Next, we have that $\dim(V_j) = 1$ or $D - 1$ for every $j = 1, \ldots, k - 1$ and in particular for $j = 1$. This means that the JH-series for $\varphi(L_0)$ is $\{0\} \subset V_1 \subset \mathbb{R}^D$, with V_1 being either of dimension or codimension 1. Again, by postcomposing φ with the transpose inverse automorphism of $\mathrm{GL}_D\mathbb{R}$ if necessary, we can assume that the 1-eigen space of L_0 is $D - 1$ dimensional.

Consider the n distinct conjugates of L that correspond to the distinct columns of $\mathrm{SL}_n(\mathfrak{O})$. By taking these conjugates of L_0, we generate $\mathrm{EL}_n(\pi^m \mathfrak{O})$. Each of these column spaces has a $D - 1$ dimensional 1-eigenspace, let us call them $W_1, \ldots, W_n$. Then, $\overset{n}{\underset{i=1}{\cap}} W_i$ is a 1-eigenspace for $\mathrm{EL}_n(\pi^m \mathfrak{O})$. The following shows that it is not trivial:

Lemma 3.4. *Let $W_1, W_2, \ldots, W_n$ be co-dimension one subspaces in a D dimensional space. Then $\dim(\cap_{i=1}^n W_i) \geqslant D - n$.*

Proof. This result follows by $\dim(W_1 \cup W_2) = \dim(W_1) + \dim(W_2) - \dim(W_1 \cap W_2)$. ✺

Let us pass to a finite index subgroup of $\mathrm{EL}_n(\pi^m \mathfrak{O})$ which is normal in $\mathrm{SL}_n(\mathfrak{O})$. Then V, the 1-eigenspace for the image of this subgroup is at least $(D - n)$-dimensional, at most $(D - 1)$-dimensional and $\mathrm{SL}_n(\mathfrak{O})$-invariant.

This gives a map $\varphi(\mathrm{SL}_n(\mathfrak{O})) \to \mathrm{GL}(V) \times \mathrm{GL}(\mathbb{R}^D/V)$. If $D = 2n - 2$, then the dimension and co-dimension of V are both less than $2n - 2$. We have already established that this means that the image of $\mathrm{SL}_n(\mathfrak{O})$ in $\mathrm{GL}(V) \times \mathrm{GL}(\mathbb{R}^D/V)$ is finite. Again, the kernel $\mathrm{stab}(V) \to \mathrm{GL}(V) \times \mathrm{GL}(\mathbb{R}^D/V)$ is abelian, which means that $\varphi(\mathrm{SL}_n(\mathfrak{O}))$ is finite.

For the case when $D = 2n - 1$, we have, once more that the dimension and co-dimension of V is less than $2n - 1$. The proof finishes as above.

4. Algebraic Facts

Let $\mathfrak{O}$ denote a local principal ideal ring. The unique maximal ideal of $\mathfrak{O}$ is denoted by $\pi\mathfrak{O}$.

Lemma 4.1. *An additive subgroup of $\mathfrak{O}$ is of finite index if and only if it contains a subgroup of the form $\pi^k \mathfrak{O}$.*

Proof. Let A be a finite index subgroup of $\mathfrak{O}$. Using the fact that $\mathfrak{O}/A$ is finite, we obtain that A is open and therefore it is also closed. The ring $\mathfrak{O}$ is a finite extension of $\mathbb{Z}_p$, so there exists $x_1, x_2, \ldots, x_g \in \mathfrak{O}$ which are algebraic over $\mathbb{Z}_p$ so that $\mathfrak{O} = \mathbb{Z}_p[x_1, \ldots, x_g]$. Since A is a finite index additive subgroup there exists elements $m_i \in \mathbb{N}$ such that $m_i x_i \in A$ for all $1 \leqslant i \leqslant g$. Let

$$m = \min\{m_i \colon m_i x_i \in A \text{ for all } i = 1, \ldots, g\}.$$

Then the cyclic group $\mathbb{Z} < mx_i >$ is contained in A, for each i. Since A is closed, we obtain that $\mathbb{Z}_p < mx_i >$ is also a subgroup of A. Therefore, we have that $m\mathbb{Z}_p[x_1, \ldots, x_g] = m\mathfrak{O}$ is contained in A.

Let $k = \mathrm{val}(M)$. Then, $\pi^k \mathfrak{O} = m\mathfrak{O}$ and so, $\pi^k \mathfrak{O} \subset A$.

<div align="right">❈</div>

Lemma 4.2. *(Generalized Hensel's Lemma) Let R be a ring that is complete with respect to the ideal $\pi\mathfrak{O}$, and let $f(x) \in R[x]$ be a polynomial. If there exists $a \in \mathfrak{O}$ such that*

$$f(a) \equiv 0 (\mathrm{mod} f'(a)^2 \pi \mathfrak{O}),$$

then there exists $a_0 \in \mathfrak{O}$ satisfying

$$f(a_0) = 0 \text{ and } a_0 \equiv a (\mathrm{mod} f'(a)\pi\mathfrak{O}).$$

If $f'(a)$ is a non zero divisor in R, then a_0 is unique.

For proof see [NZM91, Theorem 2.24]

Lemma 4.3. *For any $\mathfrak{O}$ with zero characeristic, there is a positive integer r and an element $q \in \mathfrak{O}^{\star}$ so that $q^4 = -r$.*

Proof. First we prove this result for $\mathbb{Z}_p$. For $p = 2$, consider the polynomial $f(x) = t^4 + 31$. Then for $a = 1$ we have $f(a) \equiv 0 \, \mathrm{mod}(f'(a)^2 \pi \mathfrak{O})$, where $\pi\mathfrak{O}$ denotes the maximal ideal of $\mathbb{Z}_2$. Then by 4.2, we obtain that -31 has a fourth root in $\mathbb{Z}_2$.

For $p \neq 2$, consider the polynomial $t^4 + (p - 1) \in \mathbb{Z}[t]$. This has a root in F_p, namely, $t = p - 1 \mod p$ and hence in $\mathbb{Z}_p^*$ again by Lemma 4.2. Observe that any complete discrete valuation ring $\mathfrak{O}$ is a finite extension of $\mathbb{Z}_p$ for some p, such that $\mathbb{Z}_p^{\star} \subseteq \mathfrak{O}^{\star}$. We obtain our result.   ❈

Lemma 4.4. *The group $\mathrm{SL}_n(\mathfrak{O})$ is generated by elementary unipotents.*

Proof. This is classical in the case $n \geqslant 3$ [HO89]. Assume $n = 2$. We begin by observing that this permutation matrix can be expressed as a product of elementary unipotents:

$$\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix}.$$

Let $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2\mathfrak{O}$. Then, since $ad - bc = 1$, one of the entries among a or b must have 0-valuation. Multiplying by the permutation matrix above if necessary, we may assume that a has 0-valuation.

Let us multiply $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ on the left by $\begin{pmatrix} 1 & 0 \\ -a^{-1}c & 1 \end{pmatrix}$ and on the right by $\begin{pmatrix} 1 & 0 \\ -a^{-1}b & 1 \end{pmatrix}$.

The result is a diagonal matrix which we denote by $\begin{pmatrix} a_1 & 0 \\ 0 & a_1^{-1} \end{pmatrix}$.

Finally, we see that diagonal matrices are also products of elementary unipotents:

$$\begin{pmatrix} a_1^{-1} & 0 \\ 0 & a_1 \end{pmatrix} = \begin{pmatrix} 1 & -a_1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ a_1 - 1 & 1 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ a_1^{-1} - 1 & 1 \end{pmatrix} \begin{pmatrix} 1 & -a_1(1 - a_1^2) \\ 0 & 1 \end{pmatrix}$$

❀

For any unital ring A and ideal $\mathcal{I} \subseteq A$, let $\mathrm{EL}_n(\mathcal{I})$ be the subgroup of $\mathrm{SL}_n(A)$ generated by elementary unipotents with non-trivial entries in $\mathcal{I}$.

Lemma 4.5. *The subgroup $\mathrm{EL}_n(\pi^k\mathfrak{O})$ is of finite index in $\mathrm{SL}_n(\mathfrak{O})$.*

To prove this, it is shown that $\mathrm{EL}_n(\pi^k\mathfrak{O})$ is a finite index subgroup of a congruence subgroup. Then, the proof is finished by recalling that congruence subgroups are of finite index in $\mathrm{SL}_n(\mathfrak{O})$. The following claim will be used later as well:

Claim 4.6. *The subgroup $\mathrm{EL}_2(\pi^k\mathfrak{O})$ contains the matrices of the form*

$$\begin{pmatrix} 1 + \pi^{2k}x & 0 \\ 0 & (1 + \pi^{2k}x)^{-1} \end{pmatrix}, \quad x \in \mathfrak{O}$$

Proof. This claim follows by observing that

$$\begin{pmatrix} 1 & \pi^k x \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ \pi^k y & 1 \end{pmatrix} = \begin{pmatrix} 1 + \pi^{2k}xy & \pi^k x \\ \pi^k y & 1 \end{pmatrix}$$

Further by multiplying with suitable elements of $\mathrm{EL}_2(\pi^k\mathfrak{O})$, we can bring the above matrix into the required form. ❀

Proof of 4.5. Each choice of two standard basis vectors yields an embedding of $\mathrm{EL}_2(\pi^k\mathfrak{O})$ into $\mathrm{EL}_n(\pi^k\mathfrak{O})$ which means that $\mathrm{EL}_n(\pi^k\mathfrak{O})$ at least contains diagonal matrices with two non-trivial entries of the form $1+\pi^{2k}\mathfrak{O}$. Since the diagonal group is abelian this shows that the full determinant 1-diagonal group with entries in $1 + \pi^{2k}\mathfrak{O}$ belongs to $\mathrm{EL}_n(\pi^k\mathfrak{O})$.

Now we claim that $\mathrm{EL}_n(\pi^k\mathfrak{O})$ has finite index in the congruence subgroup $\mathcal{K} = \mathrm{Ker}(\mathrm{SL}_n(\mathfrak{O}) \to \mathrm{SL}_n(\mathfrak{O}/\pi^k\mathfrak{O}))$. Every matrix of $\mathcal{K}$ is of the form $I + \pi^k(a_{ij})$ with $a_{ij} \in \mathfrak{O}$ and more specifically diagonal entries of $\mathcal{K}$ are invertible. Therefore by multiplying on the right side with suitable elements of the group $\mathrm{EL}_n(\pi^k\mathfrak{O})$ we can reduce elements of $\mathcal{K}$ to diagonal matrices. Then lemma follows by observing that $1+\pi^{2k}\mathfrak{O}$ has finite index in the multiplicative group $1+\pi^k\mathfrak{O}$ and full determinant 1-diagonal group with entries in $1 + \pi^{2k}\mathfrak{O}$ belongs to $\mathrm{EL}_n(\pi^k\mathfrak{O})$.   ✤

Corollary 4.7. *If $\rho : \mathrm{SL}_n\mathfrak{O} \to G$ is a representation so that for some $i \neq j$ the image $\rho(E_{i,j}(\mathfrak{O}))$ is finite then $\rho(\mathrm{SL}_n\mathfrak{O})$ is finite.*

Proof. If the image $\rho(E_{i,j}(\mathfrak{O}))$ is finite, then there is some k so that $E_{i,j}(\pi^k\mathfrak{O}) \leqslant \mathrm{ker}(\rho)$. This means that $\mathrm{EL}_n(\pi^k\mathfrak{O}) \leqslant \mathrm{ker}(\rho)$ and hence the kernel has finite index in $\mathrm{SL}_n\mathfrak{O}$.   ✤

Proposition 4.8. *Every finite index subgroup of $\mathrm{SL}_n(\mathfrak{O})$ has finite abelianization, i.e. it is almost perfect. Furthermore, if either $|\mathfrak{O}/\pi\mathfrak{O}| > 3$ or $n > 2$ then $\mathrm{SL}_n(\mathfrak{O})$ is perfect.*

Proof. Let $G \leqslant \mathrm{SL}_n(\mathfrak{O})$ be a finite index subgroup. Then, $G \geqslant \mathrm{EL}_n(\pi^k\mathfrak{O})$ for some k. Indeed, for each i, j the subgroup $G \cap E_{i,j}(\mathfrak{O})$ must be of finite index in $E_{i,j}(\mathfrak{O})$. Take k sufficiently large, $G \cap E_{i,j}(\mathfrak{O}) \geqslant E_{i,j}(\pi^k\mathfrak{O})$ for all i, j. Therefore, it is sufficient to show that $\mathrm{EL}_n(\pi^k\mathfrak{O})$ has finite abelianization.

For $n \geqslant 3$ this follows from the Steinberg relations which in fact shows that both $\mathrm{EL}_n(\pi^k\mathfrak{O})$ and $\mathrm{SL}_n(\mathfrak{O})$ are perfect.

For the case $n = 2$ we have the following commutator relation:

$$(2) \qquad \left[\begin{pmatrix} q & 0 \\ 0 & q^{-1} \end{pmatrix}, \begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix}\right] = \begin{pmatrix} 1 & (q^2 - 1)t \\ 0 & 1 \end{pmatrix}.$$

By Claim 4.6, we know that $\mathrm{EL}_2(\pi^k\mathfrak{O})$ contains diagonals as above for $q = 1 + \pi^{2k}y$ for $y \in \mathfrak{O}$ which give $q^2 - 1 = \pi^{4k}y'$ for some $y' \in \mathfrak{O}$. The commutator relation (2), along with its transpose analogue shows that the commutator subgroup of $\mathrm{EL}_2(\pi^k\mathfrak{O})$ contains the finite index subgroup $\mathrm{EL}_2(\pi^{4k}\mathfrak{O})$. Hence $\mathrm{EL}_2(\pi^k\mathfrak{O})$ has finite abelianisation.

Suppose that $n = 2$ and $|\mathfrak{O}/\pi\mathfrak{O}| > 3$. Given $x \in \mathfrak{O}$ we find $q \in \mathfrak{O}^*$ and $t \in \mathfrak{O}$ so that $(q^2 - 1)t = x$. This can be acheived by finding q so that $q^2 - 1$ is invertible. Recall that the kernel of the map $\mathfrak{O}^* \to (\mathfrak{O}/\pi\mathfrak{O})^*$ is given by $1 + \pi\mathfrak{O}$. Since $|\mathfrak{O}/\pi\mathfrak{O}| > 3$ we know that there is a root of unity such that $\xi^2 \notin 1 + \pi\mathfrak{O}$ and therefore $\xi^2 - 1$ is invertible. Indeed, if not, then $\xi^2 - 1 \in \pi\mathfrak{O}$. Hence $\mathrm{SL}_2(\mathfrak{O})$ is perfect in this case.   ✻

Lemma 4.9. *If $S \leqslant \mathrm{GL}_D\mathbb{R}$ is a solvable subgroup then there exists a finite index subgroup $S_0 \triangleleft S$ such that $[S_0, S_0]$ is unipotent upper-triangular.*

Proof. Let S_0 be the finite index subgroup so that the Zariski closure $\overline{S}_0^Z(\mathbb{C})$ is Zariski-connected. By the Lie-Kolchin Theorem [Hum98] $\overline{S}_0^Z(\mathbb{C})$ is conjugate into the upper triangular group and the commutator subgroup $[\overline{S}_0^Z(\mathbb{C}), \overline{S}_0^Z(\mathbb{C})]$ is unipotent. This means that $[S_0, S_0] \leqslant [\overline{S}_0^Z(\mathbb{C}), \overline{S}_0^Z(\mathbb{C})]$ is unipotent. Since the entries of S are in $\mathbb{R}$, there is an $\mathbb{R}$-basis which upper-triangulates the unipotent group $[S_0, S_0]$.   ✻

Lemma 4.10. *Let $N \leqslant \mathrm{SL}_n\mathfrak{O}$ be the upper triangular group and $U \triangleleft N$ the upper unipotent group. If N_0 is of finite index in N then $U \cap [N_0, N_0]$ has finite index in U.*

Proof. The proof is by induction on n.

For $n = 2$, we begin by observing that there exists an element $y \in \mathfrak{O}^*$ and an integer $k \geqslant 0$ such that

$$\begin{pmatrix} 1 + \pi^k y & 0 \\ 0 & (1 + \pi^k y)^{-1} \end{pmatrix} \in N_0.$$

Indeed, consider D the diagonal group. Observe that $N_0 \cap D$ is finite index in D. Therefore some power of the matrix $\begin{pmatrix} 1 + \pi & 0 \\ 0 & (1 + \pi)^{-1} \end{pmatrix}$ belongs to N_0. Let $k \in \mathbb{N}$ and $y \in \mathfrak{O}^\star$ such that $\begin{pmatrix} 1 + \pi^k y & 0 \\ 0 & (1 + \pi^k y)^{-1} \end{pmatrix} \in N_0$.

The group N_0 has finite index in N and hence $N_0 \cap E_{12}(\mathfrak{O})$ has finite index in $E_{12}(\mathfrak{O})$. using the fact that every finite index subgroup of $\mathfrak{O}$ is of the form $\pi^r\mathfrak{O}$ for some positive integer r, we can assume that $N_0 \cap E_{12}(\mathfrak{O}) = E_{12}(\pi^t\mathfrak{O})$ for some t.

For k, y as given in Lemma 2.6, observe the following commutator relation,

$$\left[\begin{pmatrix} 1 + \pi^k y & 0 \\ 0 & (1 + \pi^k y)^{-1} \end{pmatrix}, \begin{pmatrix} 1 & \pi^t x \\ 0 & 1 \end{pmatrix} \right] = \begin{pmatrix} 1 & \pi^{k+t} y x (2 + \pi^k y) \\ 0 & 1 \end{pmatrix} \in [N_0, N_0] \cap U.$$

But $\pi^{k+t} yx(2 + \pi^k y) = \pi^{r'} y$ for some r'. Since y is a unit, by Lemma 4.5 we obtain that $E_{12}(\pi^{r'} y) \subseteq [N_0, N_0] \cap U$. This prove that $[N_0, N_0] \cap U$ has finite index in U.

Assume it is true for n and let us show it for $n+1$. Consider N', U' and $[N_0', N_0']$ the restrictions of each of these to the first n-columns. Then, $[N_0', N_0'] \cap U' \geqslant U'(\pi^k \mathfrak{O})$ for some $k \geqslant 0$. Then, observing that $[N_0, N_0] \cap U$ is normal in U the commutator relation

$$[E_{i,n-1}(\pi^k \mathfrak{O}), E_{n-1,n}(\mathfrak{O})] = E_{i,n}(\pi^k \mathfrak{O})$$

gives the desired result.   ❋

5. Unanswered Questions

Question. *Are the images* $\mathrm{SL}_n \mathfrak{O} \to \mathrm{GL}_{D < 2n} \mathbb{R}$ *always trivial?*

The answer to this question in case $n = 2$ is yes, if $p \neq 2, 3$ and no, otherwise:

Corollary 5.1. *Assume that* $\mathfrak{O} \neq \mathbb{Z}_2, \mathbb{Z}_3$. *The image of any representation* $\mathrm{SL}_2 \mathfrak{O} \to \mathrm{GL}_2 \mathbb{R}$ *is trivial.*

Proof. Theorem 1.1 shows that the image of any representation $\mathrm{SL}_2 \mathfrak{O} \to \mathrm{GL}_2 \mathbb{R}$ is finite, therefore compact, and hence contained in a conjugate of the maximal compact subgroup $SO_2(\mathbb{R})$. Since $SO_2(\mathbb{R})$ is abelian and $\mathrm{SL}_2 \mathfrak{O}$ is perfect whenever $\mathfrak{O} \neq \mathbb{Z}_2, \mathbb{Z}_3$, we conclude that the image is trivial.

Also, the proof of Theorem 1.1 shows that any representation $\mathrm{SL}_2 \mathfrak{O} \to \mathrm{GL}_2 \mathbb{R}$ is necessarily reducible. This, in combination with the previous paragraph, shows that the image must be unipotent and abelian and hence trivial.   ❋

Lemma 5.2. *If* $p = 2, 3$ *then there is a non-trivial homomorphism* $\mathrm{SL}_2 \mathbb{Z}_p \to \mathrm{GL}_D \mathbb{R}$ *for every* D.

Proof. Assume $p = 2, 3$. Recall that $q = 3, 2$ is a unit in $\mathbb{Z}_2$ and $\mathbb{Z}_3$ respectively. Consider the commutator relation for this q:

$$(3) \qquad \left[\begin{pmatrix} q & 0 \\ 0 & q^{-1} \end{pmatrix}, \begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix} \right] = \begin{pmatrix} 1 & (q^2 - 1)t \\ 0 & 1 \end{pmatrix}.$$

This means that, if $(p, k) = (2, 3)$ or $(3, 1)$ then $\mathrm{EL}_2(p^k \mathbb{Z}_p) \leqslant [\mathrm{SL}_2 \mathbb{Z}_p, \mathrm{SL}_2 \mathbb{Z}_p]$. Furthermore, the group $\mathrm{EL}_2(p^k \mathbb{Z}_p)$ contains the kernel of the map $\mathrm{SL}_2(\mathbb{Z}_p) \to \mathrm{SL}_2(\mathbb{Z}/p^{2k})$. Therefore, the abelianization of $\mathrm{SL}_2 \mathbb{Z}_p$ is the abelianization of $\mathrm{SL}_2(\mathbb{Z}/p^{2k})$.

Let A be abelian quotient of $\mathrm{SL}_2(\mathbb{Z}/p^{2k})$. Then, any non-trivial representation of A into $\mathrm{GL}_D \mathbb{R}$ (of which there are many) gives a non-trivial representation of $\mathrm{SL}_2 \mathbb{Z}_p$.   ❋

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